ALMOST QUARTER-PINCHED KÄHLER METRICS
AND CHERN NUMBERS

MARTIN DERAUX AND HARISH SESHADRI

ABSTRACT. Given $n \in \mathbb{Z}^+$ and $\varepsilon > 0$, we prove that there exists $\delta = \delta(\varepsilon, n) > 0$ such that the following holds: If (M^n, g) is a compact Kähler n-manifold whose sectional curvatures K satisfy

$$-1 - \delta \leq K \leq -\frac{1}{4},$$

and $c_I(M)$, $c_J(M)$ are any two Chern numbers of M, then

$$\left| \frac{c_I(M)}{c_J(M)} - \frac{c_I^0}{c_J^0} \right| < \varepsilon,$$

where c_I^0, c_J^0 are the corresponding characteristic numbers of a complex-hyperbolic space-form.

It follows that the Mostow-Siu surfaces and the threefolds of Deraux do not admit Kähler metrics with pinching close to $\frac{1}{4}$.

1. INTRODUCTION

It is well known that Riemannian manifolds of constant sectional curvature -1 must be locally isometric to real hyperbolic space. Allowing the sectional curvatures to be merely close to -1 (say with pinching close to 1, i.e. with a suitable normalization all sectional curvatures are between $-\alpha$ and -1 for some α close to 1), there are many more examples than hyperbolic manifolds.

In fact, in [3], Gromov and Thurston construct manifolds of dimension ≥ 4 that admit negatively curved Riemannian metrics with pinching arbitrarily close to 1, but that do not admit any metric of constant curvature (in fact they do not admit any locally symmetric metric).

The Kähler analogue of the above examples is not well understood. Negatively curved Kähler metrics must have pinching at most $\frac{1}{4}$, as can be deduced from an inequality due to Berger (see [1]), and $\frac{1}{4}$-pinched metrics are precisely those locally isometric to complex hyperbolic space (of dimension ≥ 2). It is not known, however, whether there exist negatively curved Kähler manifolds with pinching arbitrarily close to $\frac{1}{4}$ which are not biholomorphic to ball quotients.

The surfaces constructed by Mostow and Siu in [7] (as well as their three-dimensional analogue, see [2]) admit Kähler metrics with negative sectional curvatures, but very little is known about the pinching of these metrics.

The main result of this note relates proximity to $\frac{1}{4}$-pinching with proximity of the ratios of Chern numbers to the corresponding ratios of Chern numbers of a complex hyperbolic manifold of the same dimension. This shows that neither the Mostow-Siu surfaces, nor the three-dimensional analogues constructed in [2], admit Kähler metric with pinching close to $\frac{1}{4}$. This answers a question raised by Pansu

in [8] (echoing a question of Gromov), which seems not to be answered anywhere in the literature.

2. Pinching and ratios of Chern numbers

The main observation of this note is the following:

Theorem 2.1. *Given $n \in \mathbb{Z}^+$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, n) > 0$ such that the following holds: If (M^n, g) is a compact Kähler n-manifold whose sectional curvatures K satisfy*

$$-1 - \delta \leq K \leq -\frac{1}{4},$$

and $c_I(M)$, $c_J(M)$ are any two Chern numbers of M, then

$$\left| \frac{c_I(M)}{c_J(M)} - \frac{c_I^0}{c_J^0} \right| < \varepsilon,$$

where c_I^0, c_J^0 are the corresponding characteristic numbers of a complex-hyperbolic space-form.

Recall that Mostow and Siu constructed an *infinite* family of surfaces that admit negatively curved Kähler metrics, but the corresponding ratios c_1^2/c_2 do not approach 3 as one ranges through these examples (see the introduction of [7]). The ratios c_1^3/c_3 for the threefolds constructed in [2] certainly cannot approach 16, since only a finite number of threefolds were constructed there.

Theorem 2.1 implies that the Mostow-Siu surfaces (as well as the threefolds in [2]) do not carry Kähler metrics with pinching close to 1/4.

It is an interesting open question whether the conclusion of Theorem 2.1 holds if only assumes the existence of almost-quarter pinched *Riemannian* metrics on a compact Kähler manifold. This question is natural in light of the result of Hernandez-Zheng-Yau ([4], [9]): If a compact Kähler manifold M admits a negatively quarter-pinched Riemannian g metric, then (M, g) is holomorphically isometric to a complex-hyperbolic space-form.

3. proof

Let V be a vector space over $\mathbb{R}$ of real dimension $2n$ with an almost complex structure $J : V \to V$ and a J invariant inner product $\langle \ , \ \rangle$. Hence $J^2 = -I$ and $\langle J(v), J(w) \rangle = \langle v, w \rangle$ for all $v, w \in V$.

A *curvature tensor* on V is a 4-linear map $R : V \times V \times V \times V \to \mathbb{R}$, i.e. an element of $\otimes^4 V^*$, satisfying

(1) $R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z)$
(2) $R(Z, W, X, Y) = R(X, Y, Z, W)$
(3) $R(X, Y, Z, W) + R(X, W, Y, Z) + R(X, Z, W, Y) = 0.$

If, in addition, R satisfies

(4) $R(J(X), J(Y), Z, W) = R(X, Y, J(Z), J(W)) = R(X, Y, Z, W)$

then we say that R is a *Kähler curvature tensor*.

Let $\mathcal{R} \subset \otimes^4 V^*$ denote the space of Kähler curvature tensors and $R_0 \in \mathcal{R}$ denote the curvature tensor of complex hyperbolic space $\mathbb{C}H^n$.

R_0 is given by

$$(3.1) \qquad -4R_0(u, v, z, w) = \langle u, z \rangle \langle v, w \rangle - \langle u, w \rangle \langle v, z \rangle + \langle u, J(z) \rangle \langle v, J(w) \rangle$$
$$- \langle u, J(w) \rangle \langle v, J(z) \rangle + 2 \langle u, J(v) \rangle \langle z, J(w) \rangle$$

It is well-known that if R is a Kähler curvature tensor on V whose sectional curvatures are negative and $\frac{1}{4}$-pinched, then $R = R_0$. We generalize this in Proposition 3.2. Recall that we regard the space $\mathcal{R}$ of Kähler curvature tensors as a subspace of $\otimes^4 V^*$. The inner product on V canonically induces an inner product on $\otimes^4 V^*$ and hence on $\mathcal{R}$.

In what follows, we will use the following notation: For arbitrary $u, v \in V$,

$$K(u, v) := R(u, v, u, v).$$

If u and v are orthornormal, this is the sectional curvature of the 2-plane spanned by u and v. For any u, let

$$H(u) := R(u, J(u), u, J(u)).$$

If u is a unit vector this is the holomorphic sectional corresponding to u. Note that $H(u) = -1$ for all unit vectors u if $R = R_0$.

Lemma 3.1. *Given $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon)$ such that if R is a Kähler curvature tensor with $|H(u) + 1| < \eta$ for all unit vectors u, then $|R - R_0| < \varepsilon$.*

Proof. Note that $|R - R_0| < \varepsilon$ if $|R(x, y, z, t) - R_0(x, y, z, t)| < \frac{\varepsilon}{(2n)^4}$ for every orthonormal set $\{x, y, z, t\}$.

Let $u, v \in V$ be unit vectors. If a, b are real numbers such that $a^2 + b^2 = 1$, then (this calculation is from [5], Lemma 3)

$$H(au + bv) + H(au - bv) = 2a^4 H(u) + 2b^4 H(v)$$
$$+ 12a^2 b^2 R(u, J(u), v, J(v)) - 8a^2 b^2 R(u, v, u, v)$$

$$H(au + bJ(v)) + H(au - bJ(v)) = 2a^4 H(u) + 2b^4 H(v)$$
$$+ 12a^2 b^2 R(u, J(u), v, J(v)) - a^2 b^2 R(u, J(v), u, J(v))$$

We also have the equation

$$R(u, v, u, v) + R(u, J(v), u, J(v)) - R(u, J(u), v, J(v)) = 0,$$

which follows from (3) and (4) in the definition of a Kähler curvature tensor. Take $a = b = \frac{1}{\sqrt{2}}$. It is clear that we can use these three equations to express $R(u, v, u, v)$ as a linear combination of

$$H(u), H(v), H(\frac{u+v}{\sqrt{2}}), H(\frac{u-v}{\sqrt{2}}), H(\frac{u+J(v)}{\sqrt{2}}), H(\frac{u-J(v)}{\sqrt{2}}).$$

Moreover note that the coefficients in this expression are absolute constants, not dependent on anything.

On the other hand, one knows that the full curvature tensor can be expressed in terms of sectional curvatures. In fact [6],

$$24R(x, y, z, t) = K(x + z, y + t) + K(x - z, y - t) - K(x + z, y - t) - K(x - z, y + t)$$
$$- K(x + t, y + z) - K(x - t, y - z) + K(x + t, y - z) + K(x - t, y + z)$$

It is clear from this expression and the earlier one expressing $K(x,y)$ in terms of holomorphic sectional curvatures that $|R(x,y,z,t) - R_0(x,y,z,t)| < \frac{\varepsilon}{(2n)^4}$ if $|H(u) + 1| < \eta$ for an $\eta = \eta(\varepsilon, n)$.

$\square$

Proposition 3.2. *Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, n)$ such that if the sectional curvatures of a Kähler curvature tensor $R \in \mathcal{R}$ satisfy*

$$-1 - \delta \ \leq \ K \ \leq \ -\frac{1}{4}$$

then

$$|R - R_0| < \varepsilon$$

in the norm on $\otimes^4 V^*$.

Proof. By the lemma above, it is enough to show that there is a $\delta = \delta(\eta) > 0$ such that $-1 - \delta \ \leq \ K \ \leq \ -\frac{1}{4}$ implies $|H(u) + 1| < \eta$ for all unit vectors u.

Let u be any unit vector in V. Choose v so that $\{u, J(u), v, J(v)\}$ is an orthonormal set. By (3) and (4) in the definition of a Kähler curvature tensor,

(3.2) $\qquad K(u,v) + K(u, J(v)) - R(u, J(u), v, J(v)) = 0.$

We recall the following estimate of Berger ([1], Page 69, (7)). Note that though Berger works with positive pinching, his proof works for negative pinching as well. Also, our pinching hypothesis and the conclusion can be obtained from his by a rescaling. If R is a curvature tensor (not necessarily Kähler) which is α-pinched, i.e. $-\alpha \leq K \leq -\frac{1}{4}$ then

$$|R(X,Y,Z,W)| \leq \frac{2}{3}(\alpha - \frac{1}{4}).$$

for any orthonormal set $\{X, Y, Z, W\}$.

Let $\eta > 0$ be as in Lemma 3.2. By Berger's estimate there exists $\delta_1 = \delta_1(\eta)$ such that if the sectional curvatures K satisfy

(3.3) $\qquad -1 - \delta_1 \ \leq \ K \ \leq \ -\frac{1}{4}$

then

(3.4) $\qquad |R(u, J(u), v, J(v))| \leq \frac{1}{2} + \frac{\eta}{6}.$

By (3.2) and (3.4)

$$K(u,v) \ = \ -K(u, J(v)) \ + \ R(u, J(u), v, J(v))$$
$$\geq \ \frac{1}{4} \ - \ \frac{1}{2} \ - \frac{\eta}{6} \ = \ -\frac{1}{4} \ - \frac{\eta}{6}$$

(3.5)

Of course, we have the same estimates for $K(u, J(v))$, $K(J(u), v))$ and $K(J(u), J(v))$. Hence,

(3.6) $\qquad -\frac{1}{4} - \frac{\eta}{6} \ \leq \ c \ \leq \ -\frac{1}{4}, \qquad -\frac{1}{2} - \frac{\eta}{3} < R(u, J(u), v, J(v)) \leq -\frac{1}{2},$

where

$$c \in \{K(u,v), \ K(u, J(v)), \ K(J(u), v), \ K(J(u), J(v))\}.$$

Applying the second inequality in (3.6) to the orthonormal set

$$\{\frac{1}{\sqrt{2}}(u - J(v)), \ \frac{1}{\sqrt{2}}(J(u) + v), \ \frac{1}{\sqrt{2}}(u + J(v)), \ \frac{1}{\sqrt{2}}(J(u) - v)\},$$

we get

(3.7) $$R(u - J(v), \ J(u) + v, \ u + J(v), \ J(u) - v) \le -2.$$

Expanding the left hand side

(3.8) $$R(u - J(v), \ J(u) + v, \ u + J(v), \ J(u) - v) =$$

$$H(u) + H(v) + 2R(u, J(u), v, J(v)) - 4K(u, v)$$

Combining (3.6) and (3.7) we have

(3.9) $$H(u) \ + \ H(v) \ - \ 4K(u, v) \ < \ -1 + \frac{2\eta}{3}.$$

(3.9) and the first inequality in (3.6) imply

$$H(u) \ + \ H(v) \ < \ -2 + \frac{2\eta}{3}.$$

Finally, using (3.3), the above inequality implies

$$-1 - \delta_1 \ < \ H(u) \ < \ -1 + \frac{2\eta}{3} + \delta_1.$$

Let

$$\delta = min\{\frac{\eta}{3}, \delta_1\}.$$

Then

$$|H(u) + 1| \ < \ \eta$$

and the proof is complete since the unit vector u was arbitrary.

$$\square$$

Let ω denote the Kähler form of V, defined by $\omega(u, v) = \langle u, J(v) \rangle$ Then the volume form on V, with V carrying the natural orientation coming from J, is just ω^n.

Let $I = (a_1, ..., a_n)$ and $J = (b_1, ..., b_n)$ be muti-indices in $(\mathbb{Z}_{\ge 0})^n$ with $\Sigma_i i a_i = \Sigma_i i b_i = n$. Given an Kähler curvature tensor R we can form the corresponding Chern forms $c_I(R) = c_1^{a_1}(R) \wedge ... \wedge c_n^{a_n}(R)$ and $c_J(R) = c_1^{b_1}(R) \wedge ... \wedge c_n^{b_n}(R)$. These are elements of $\wedge^n V^*$.

Given ε_1 (to be chosen below), Proposition (3.2) (and the obvious continuous dependence of the Chern forms on the full curvature tensor) implies there is a $\delta = \delta(\varepsilon_1, n)$ such that $-1 - \delta < K < -1$ implies that

(3.10) $$|c_I(R) - c_I(R_0)| < \varepsilon_1, \quad |c_J(R) - c_J(R_0)| < \varepsilon_1$$

in the norm on $\wedge^n V^*$.

Let $c_I(R_0) = a(n)\omega^n$ and $c_J(R_0) = b(n)\omega^n$. Choose $\varepsilon_1 = \varepsilon_1(\varepsilon, n)$ so that

(3.11) $$\frac{a(n) - \varepsilon_1}{b(n) + \varepsilon_1} > \frac{a(n)}{b(n)} - \varepsilon \qquad \frac{a(n) + \varepsilon_1}{b(n) - \varepsilon_1} < \frac{a(n)}{b(n)} + \varepsilon$$

Now (3.10) implies

$$(a(n) - \varepsilon_1)\omega^n \ < \ c_I(R) \ < \ (a(n) + \varepsilon_1)\omega^n$$

and

$$(b(n) - \varepsilon_1)\omega^n \;\; < \;\; c_J(R) \;\; < \;\; (b(n) + \varepsilon_1)\omega^n$$

We now turn to a compact Kähler manifold with almost-quarter pinching as in Theorem 1.1. We can integrate the above inequalities on M to get

$$(a(n) - \varepsilon_1)Vol(M) \;\; < \;\; c_I(M) \;\; < \;\; (a(n) + \varepsilon_1)Vol(M)$$

and

$$(b(n) - \varepsilon_1)Vol(M) \;\; < \;\; c_J(M) \;\; < \;\; (b(n) + \varepsilon_1)Vol(M).$$

Dividing these two inequalities and using (3.11) gives the desired result.

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INSTITUT FOURIER, UNIVERSITÉ DE GRENOBLE I, 38402 SAINT-MARTIN-D'HÈRES CEDEX, FRANCE
E-mail address: `deraux@ujf-grenoble.fr`

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560012, INDIA
E-mail address: `harish@math.iisc.ernet.in`